

GENERALIZED LCM MATRICES

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ABSTRACT. Let f be an arithmetical function. The matrix $[f(i, j)]_{n \times n}$ given by the value of f in least common multiple of $[i, j]$, $f([i, j])$ as its i, j entry is called the least common multiple (LCM) matrix. We consider the generalization of this matrix where the elements are in the form $f(n, [i, j])$ and $f(n, i, j, [i, j])$.

1. INTRODUCTION

The classical Smith determinant was introduced in 1875 by H. J. S. Smith [12] who also proved that

$$(1) \quad \det[(i, j)]_{n \times n} = \begin{vmatrix} (1, 1) & (1, 2) & \cdots & (1, n) \\ (2, 1) & (2, 2) & \cdots & (2, n) \\ \cdots & \cdots & \cdots & \cdots \\ (n, 1) & (n, 2) & \cdots & (n, n) \end{vmatrix} = \varphi(1)\varphi(2) \cdots \varphi(n),$$

where (i, j) represents the greatest common divisor of i and j , and $\varphi(n)$ denotes the Euler totient function.

The GCD matrix with respect to f is

$$[f(i, j)]_{n \times n} = \begin{bmatrix} f((1, 1)) & f((1, 2)) & \cdots & f((1, n)) \\ f((2, 1)) & f((2, 2)) & \cdots & f((2, n)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n, 1)) & f((n, 2)) & \cdots & f((n, n)) \end{bmatrix}.$$

There are quite a few generalized forms of GCD matrices, which can be found in several references [1, 3, 7, 8, 11].

H. J. S. Smith [12] also evaluated the determinant of

$$[[i, j]]_{n \times n} = \begin{bmatrix} [1, 1] & [1, 2] & \cdots & [1, n] \\ [2, 1] & [2, 2] & \cdots & [2, n] \\ \cdots & \cdots & \cdots & \cdots \\ [n, 1] & [n, 2] & \cdots & [n, n] \end{bmatrix},$$

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and proved that

$$\begin{aligned} \det [i, j]_{n \times n} &= (n!)^2 g(1)g(2) \cdots g(n) = \\ &= \prod_{k=1}^N \varphi(k) \prod_{p|k} (-p). \end{aligned}$$

where $g(n) = \frac{1}{n} \sum_{d|n} d\mu(d)$, $\mu(n)$ being the classical Möbius function. The structure of an LCM matrix $[i, j]_{n \times n}$ is the following (I. Korkee, P. Haukkanen [10])

$$[i, j]_{n \times n} = \text{diag}(1, 2, \dots, n) A A^T \text{diag}(1, 2, \dots, n),$$

where $A = [a_{ij}]_{n \times n}$,

$$a_{ij} = \begin{cases} \sqrt{g(j)}, & \text{if } j | i \\ 0, & \text{if } j \nmid i \end{cases}.$$

The LCM matrix with respect to f is

$$[f[i, j]]_{n \times n} = \begin{bmatrix} f([1, 1]) & f([1, 2]) & \cdots & f([1, n]) \\ f([2, 1]) & f([2, 2]) & \cdots & f([2, n]) \\ \cdots & \cdots & \cdots & \cdots \\ f([n, 1]) & f([n, 2]) & \cdots & f([n, n]) \end{bmatrix}.$$

Results concerning LCM matrices appear in papers S. Beslin [2], K. Bourque, S. Ligh [4], W. Feng, S. Hong, J. Zhao [6] P. Haukkanen, J. Wang and J. Sillanpää [7].

In this paper we study matrices which have as variables the least common multiple and the indices

$$[f(n, [i, j])]_{n \times n} = \begin{bmatrix} f(n, [1, 1]) & f(n, [1, 2]) & \cdots & f(n, [1, n]) \\ f(n, [2, 1]) & f(n, [2, 2]) & \cdots & f(n, [2, n]) \\ \cdots & \cdots & \cdots & \cdots \\ f(n, [n, 1]) & f(n, [n, 2]) & \cdots & f(n, [n, n]) \end{bmatrix}$$

and the more general form matrices

$$[f(n, i, j, [i, j])]_{n \times n} = \begin{bmatrix} f(n, 1, 1, [1, 1]) & f(n, 1, 2, [1, 2]) & \cdots & f(n, 1, n, [1, n]) \\ f(n, 2, 1, [2, 1]) & f(n, 2, 2, [2, 2]) & \cdots & f(n, 2, n, [2, n]) \\ \cdots & \cdots & \cdots & \cdots \\ f(n, n, 1, [n, 1]) & f(n, n, 2, [n, 2]) & \cdots & f(n, n, n, [n, n]) \end{bmatrix}$$

2. GENERALIZED LCM MATRICES

Theorem 2.1. For a given totally multiplicative aritmetical function $g(n)$ let

$$f(n, [i, j]) = g([i, j]) \sum_{k \leq \frac{n}{[i, j]}} g(k).$$

Then

$$(2) \quad [f(n, [i, j])]_{n \times n} = C_n^T \text{diag}(g(1), g(2), \dots, g(n)) C_n,$$

where $C_n = [c_{ij}]_{n \times n}$

$$c_{ij} = \begin{cases} 1, & \text{if } j \mid i \\ 0, & \text{if } j \nmid i \end{cases}.$$

For a determinant we have

$$(3) \quad \det [f(n, [i, j])]_{n \times n} = g(1)g(2) \cdots g(n).$$

Proof. After multiplication, the general element of $A = (a_{ij})_{n \times n}$,

$$A = C_n^T \text{diag}(g(1), g(2), \dots, g(n)) C_n$$

is

$$a_{ij} = \sum_{k=1}^n c_{ki} g(k) c_{kj} = \sum_{\substack{i \mid k \\ j \mid k \\ k \leq n}} g(k) = \sum_{\substack{[i, j] \mid k \\ k \leq n}} g(k) = \sum_{l \leq \frac{n}{[i, j]}} g([i, j]l).$$

Because $g(n)$ is totally multiplicative

$$a_{ij} = g([i, j]) \sum_{l \leq \frac{n}{[i, j]}} g(l) = f([i, j]).$$

If we calculate the determinant of both parts of (2) we have (3). \square

Particular cases

Example 1. If $g(n) = 1$, then

$$f(n, [i, j]) = \left\lfloor \frac{n}{[i, j]} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x .

From Theorem 2.1 we have

$$\begin{aligned} \left[\left\lfloor \frac{n}{[i, j]} \right\rfloor \right]_{n \times n} &= C_n^T \text{diag}(1, 1, \dots, 1) C_n, \\ \det \left[\left\lfloor \frac{n}{[i, j]} \right\rfloor \right]_{n \times n} &= 1. \end{aligned}$$

Example 2. If $g(n) = n$, then

$$f(n, [i, j]) = \frac{\left\lfloor \frac{n}{[i, j]} \right\rfloor \left\lfloor \frac{n}{[i, j]} + 1 \right\rfloor}{2}.$$

The decomposition of generalized LCM matrix is

$$[f(n, [i, j])]_{n \times n} = C_n^T \text{diag}(1, 2, \dots, n) C_n,$$

and the determinant

$$\det [f(n, [i, j])]_{n \times n} = n!.$$

Example 3. If $g(n) = (-1)^{\Omega(n)}$ is a Liouville function, then

$$f(n, [i, j]) = (-1)^{\Omega([i, j])} \sum_{k \leq \frac{n}{[i, j]}} (-1)^{\Omega(k)}$$

and

$$[f(n, [i, j])]_{n \times n} = C_n^T \text{diag}(1, -1, \dots, (-1)^{\Omega(n)}) C_n,$$

$$\det [f(n, [i, j])]_{n \times n} = (-1)^{\sum_{k=1}^n \Omega(k)}.$$

We remark that matrices related to the greatest integer function appeared in [9, 5].

Theorem 2.2. For a given totally multiplicative function g let

$$f(n, i, j, [i, j]) = \sum_{k \leq n} g(k) - g(i) \sum_{l \leq \frac{n}{i}} g(l) - g(j) \sum_{l \leq \frac{n}{j}} g(l) + g([i, j]) \sum_{k \leq \frac{n}{[i, j]}} g(k).$$

Then

$$[f(n, i, j, [i, j])]_{n \times n} = D_n^T \text{diag}[g(1), g(2), \dots, g(n)] D_n,$$

where $D_n = [d_{ij}]_{n \times n}$,

$$d_{ij} = \begin{cases} 1, & \text{if } j \nmid i \\ 0, & \text{if } j \mid i \end{cases}.$$

Proof. After multiplication the general element of the matrix

$$A = [a_{ij}]_{n \times n} = D_n^T \text{diag}[g(1), g(2), \dots, g(n)] D_n$$

is

$$\begin{aligned}
a_{ij} &= \sum_{\substack{i \nmid k \\ j \nmid k \\ k \leq n}} g(k) = \sum_{k \leq n} g(k) - \sum_{i \mid k} g(k) - \sum_{j \mid k} g(k) + \sum_{\substack{i \mid k \\ j \mid k \\ k \leq n}} g(k) = \\
&= \sum_{k \leq n} g(k) - \sum_{il \leq n} g(il) - \sum_{jl \leq n} g(jl) + \sum_{\substack{[i,j] \mid k \\ k \leq n}} g(k)
\end{aligned}$$

The total multiplicativity of g implies,

$$\begin{aligned}
a_{ij} &= \sum_{k \leq n} g(k) - g(i) \sum_{l \leq \frac{n}{i}} g(l) - g(j) \sum_{l \leq \frac{n}{j}} g(l) + g([i, j]) \sum_{k \leq \frac{n}{[i, j]}} g(k) = \\
&= f(n, i, j, [i, j]).
\end{aligned}$$

□

Particular cases

Example 4. If $g(n) = 1$, then

$$f(n, i, j, [i, j]) = \tau(n) - \tau\left(\left\lfloor \frac{n}{i} \right\rfloor\right) - \tau\left(\left\lfloor \frac{n}{j} \right\rfloor\right) + \left\lfloor \frac{n}{[i, j]} \right\rfloor,$$

where $\tau(n) = \sum_{d \mid n} 1$. By Theorem 2.2

$$\left[f\left(n, i, j, \left\lfloor \frac{n}{[i, j]} \right\rfloor\right) \right]_{n \times n} = D_n^T \text{diag}(1, 1, \dots, 1) D_n.$$

Example 5. If $g(n) = n$, then

$$f(n, i, j, [i, j]) = \sigma(n) - \sigma\left(\left\lfloor \frac{n}{i} \right\rfloor\right) - \sigma\left(\left\lfloor \frac{n}{j} \right\rfloor\right) + \frac{\left\lfloor \frac{n}{[i, j]} \right\rfloor \left\lfloor \frac{n}{[i, j]} + 1 \right\rfloor}{2},$$

where $\sigma(n) = \sum_{d \mid n} d$.

The general form of a generalized LCM matrix is

$$[f(n, i, j, [i, j])]_{n \times n} = D_n^T \text{diag}(1, 2, \dots, n) D_n.$$

Example 6. If $g(n) = (-1)^{\Omega(n)}$ is the Liouville function then

$$\begin{aligned} f(n, i, j, [i, j]) &= \sum_{k \leq n} (-1)^{\Omega(k)} - (-1)^{\Omega(i)} \sum_{l \leq \frac{n}{i}} (-1)^{\Omega(l)} - (-1)^{\Omega(j)} \sum_{l \leq \frac{n}{j}} (-1)^{\Omega(l)} g + \\ &+ (-1)^{\Omega([i, j])} \sum_{k \leq \frac{n}{[i, j]}} (-1)^{\Omega(k)} \end{aligned}$$

and

$$[f(n, i, j, [i, j])]_{n \times n} = D_n^T \text{diag}(1, -1, \dots, (-1)^{\Omega(n)}) D_n.$$

Remark 2.1. Due to the fact that the first line of the matrix $[f(n, i, j, [i, j])]_{n \times n}$ contains only 0-s, the determinant of the matrix will always be 0.

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